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DETECTION OF A TARGET LEAVING INTERMITTENT TRACES

by

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A target reveals its presence by occasional emission of "blobs" of material that are temporarily detectable by a searcher moving in the same medium. The main question addressed is "how long will it take the searcher to detect a blob?" Diffusion of the blobs in two and three dimensions is included as a special case.

1. Summary: It sometimes happens that a physical phenomenon (target) leaves occasional traces of its existence as patches of emitted material (blobs) that may temporarily grow in size, but which ultimately dissipate and become undetectable. In this report we investigate the chances of contacting one of these blobs, assuming that the target moves in its medium enough to prevent effective overlap of one blob on another. Search is assumed to be random within some confined region by a sensor capable of detecting a given concentration of emitted material; the effect of this assumption is to make the contact rate at any time proportional to the total projected length (in two dimensions) or area (in three dimensions) of all extant blobs. The random search assumption is made for analytical convenience; nonetheless, the assumption is often a surprisingly accurate representation of real searches, particularly when the target is moving.

Let C be the mean number of times that a blob is contacted. Formulas for C are derived in sections 3 and 4 for the cases of diffusion in two and three dimensions. As long as C is much smaller than 1, the mean time $E(T)$ until the first blob is contacted is approximately an exponential random variable with mean $1/\lambda C$, where λ is the Poisson rate at which blobs are emitted. Actually, $1/\lambda C$ is a slight underestimate of $E(T)$, as is shown in sec. 2. Exact formulas for the probability distribution of T (formula A4) and $E(T)$ (formula A5) valid for all C are derived in the appendix.

The blob that is detected may be an old one if blobs have a long lifetime. The time required to detect a blob whose age does not exceed t_o could be analyzed by truncating the rate of detection function at t_o; i.e., $r(x) = 0$ for $x > t_o$. The effect would be to reduce the dimensionless constant in formulas (9) and (16).

2. The mean time to detection exceeds $1/\lambda C$

On account of the random search assumption, the total number of contacts when the blob emission times are given is a non-homogeneous Poisson process. Let $Y(t)$ be the mean value function of this process, and let $X(t)$ be the actual number of contacts up to time t. Since $Y(t)$ is itself a stochastic process, $X(t)$ is doubly stochastic [1]. Since $X(t)$ is a Poisson random variable with mean $Y(t)$, $E(X(t)|Y(t)) = Y(t)$, and $E(X(t)) = E(Y(t))$ by conditional expectations. This is handy because $E(Y(t))$ is easy to predict in equilibrium. In fact, let C be the average number of contacts per blob (see sections 3 and 4 for examples of how C can be computed). Assuming that blobs are emitted at the Poisson rate λ, we must have

$$(1) \qquad E(X(t)) = E(Y(t)) = \lambda C t$$

It follows that the average time between contacts is $1/\lambda C$. However, this is not the same thing as the mean time $E(T)$ until the first contact, since the origin of T is meant to be an arbitrary point in the process, rather than a point at which a detection has just occurred and for which $Y(t)$ is therefore likely to be large.

Since T is a positive random variable, $E(T) = \int_0^\infty P(T > t)dt$, and therefore $E(T) = E(\int_0^\infty P(T > t|Y(t))dt)$ by conditional expectations. But the event $(T > t)$ is the same as the event $(X(t) = 0)$, and $P(X(t) = 0|Y(t)) = \exp(-Y(t))$, so

$$(2) \qquad E(T) = E\left(\int_0^\infty \exp(-Y(t))dt\right) \geq \int_0^\infty \exp(-E(Y(t)))dt$$

$$= \int_0^\infty \exp(-\lambda C t)dt = 1/\lambda C ,$$

where the inequality is Jensen's inequality applied to the convex function $\exp(-x)$. So the mean time to the first contact exceeds $1/\lambda C$, in general. The condition for $E(T)$ to be approximately $1/\lambda C$ is that $Y(t)$ should not be "very random". Intuitively, this will be the case if the contribution due to a single blob is small; i.e., if $C \ll 1$. This is correct, but the proof is not trivial, so we defer it to the appendix.

3. Evaluation of C for turbulent diffusion in the ocean

In [4], it is stated that the diffusion of substance from a point in the ocean is essentially a two dimensional problem, since there is very little vertical mixing compared to turbulent horizontal processes. Several radially symmetric formulas are offered to describe the way in which the concentration of a substance $s(t,r)$ changes with distance (r) and time (t) from the point of injection, most of which are special cases of

$$(3) \qquad s(t,r) = \frac{M}{\pi(\beta t)^{2\alpha/b}\Gamma(1+2/b)} \exp\left(-r^b/(\beta t)^{\alpha}\right),$$

where M is the amount of material released, α and b are dimensionless, and β has whatever units are required to make $s(t,r)$ have units of material per unit area. Γ is the Gamma function, and is required in (3) in order to make the volume underneath $s(t,r)$ equal to M for all t. The case of classical diffusion is $\alpha = 1$, $b = 2$.

At time t after injection, a concentration sensor will be able to detect the substance within some distance $\rho(t)$ of the point at which it was released. Suppose that M units of material could just barely be detected inside a circle of radius r_0 if the material were spread uniformly; r_0 is a measure of the sensor's sensitivity. Then $\rho(t)$ must satisfy $s(t,\rho(t)) = M/(\pi r_0^2)$ at all times t. Using (3) to solve for $\rho(t)$, we obtain

$$(4)\qquad \rho(t) = (\beta t)^{\alpha/b}\ \ell n\left(\frac{r_0^2}{(\beta t)^{2\alpha/b}\Gamma(1+2/b)}\right)^{1/b},$$

except that $\rho(t)$ is taken to be 0 rather than negative. If the sensor is conducting a random search at speed v in an area of size A, then the rate of contacting the blob at time t is $2v\rho(t)/A$, since $2\rho(t)$ is the projection of the circular blob normal to the searcher's track. The average number of detections per blob is therefore

$$(5)\qquad C = \int_0^\infty (2v\rho(t)/A)dt\,.$$

Let $$K = r_0^2/\Gamma(1+2/b),$$

substitute $\rho(t)$ from (4) in (5), and then substitute $u = (\beta t)^{2\alpha/b}/K$. The result is

$$(6)\qquad C = (2v/A)\int_0^1 [Ku]^{1/2}[\ell n\ \tfrac{1}{u}]^{1/b}\left(\frac{Kb\ du}{2\alpha\beta(Ku)^{1-b/2\alpha}}\right),$$

where the upper limit of u is the largest number for which the integrand is positive. Factoring out constants and collecting powers of Ku, (6) is

$$(7)\qquad C = \frac{vKb}{A\alpha\beta}\int_0^1 [Ku]^{b/2\alpha-1/2}\ [\ell n\ \tfrac{1}{u}]^{1/b}\ du$$

The definite integral has a closed form expression [3]. Substituting this in (7), we obtain

$$(8)\qquad C = vK^{b/2\alpha+1/2}\ b\quad \Gamma(1+1/b)/[A\alpha\beta\ (b/2\alpha+1/2)^{1+1/b}]\ .$$

Recalling the definition of K, we finally have

$$C = \frac{v\, r_0^{b/\alpha+1}\, b\Gamma(1+1/b)}{A\alpha\beta\, \Gamma(1+2/b)^{b/2\alpha+1/2}\, (b/2\alpha+1/2)^{1+1/b}} . \tag{9}$$

In classical diffusion, since $\Gamma(3/2) = \sqrt{\pi}/2$ and $\Gamma(2) = 1$, (9) reduces to

$$C = \frac{v\, r_0^3 \sqrt{\pi}}{A\beta\, (3/2)^{3/2}} . \tag{10}$$

However, classical diffusion is not a good description of turbulent spreading in the ocean. According to reference [4], better descriptions would be $\alpha=2$, $b=2$, $\beta=2$ cm./sec. (Okubo and Pritchard), or $\alpha=3$, $b=2$, $\beta=.05$ $cm^{2/3}$/sec, (Okubo). Both of these models are from table 5 of reference [4]. If $v=10^4$ cm/sec., $r_0=10^3$ cm., and $A=10^{10}cm^2$, (9) gives $C = .44$ for the first of these models, and $C = 1.55$ for the second. These calculations may give the reader some idea of circumstances where $C \ll 1$. The fact that different investigators have arrived at models that provide answers differing by a factor of 3.5 should not be disturbing when the subject is turbulence in the ocean.

Figure 1 shows (4) for the same two models. The maximum blob size depends only on r_0 and b, and is consequently the same in both cases. The fact that the area under one curve is 3.5 times as large as the area under the other is essentially a question of blob lifetime.

4. Classical diffusion in three dimensions.

This derivation will parallel that of sec. 3. If $S(t,r)$ is now the concentration of material per unit volume and D the diffusion constant, then (3) is replaced by

$$S(t,r) = \frac{M}{(2\pi Dt)^{3/2}} \exp(-r^2/2Dt). \tag{11}$$

We also have

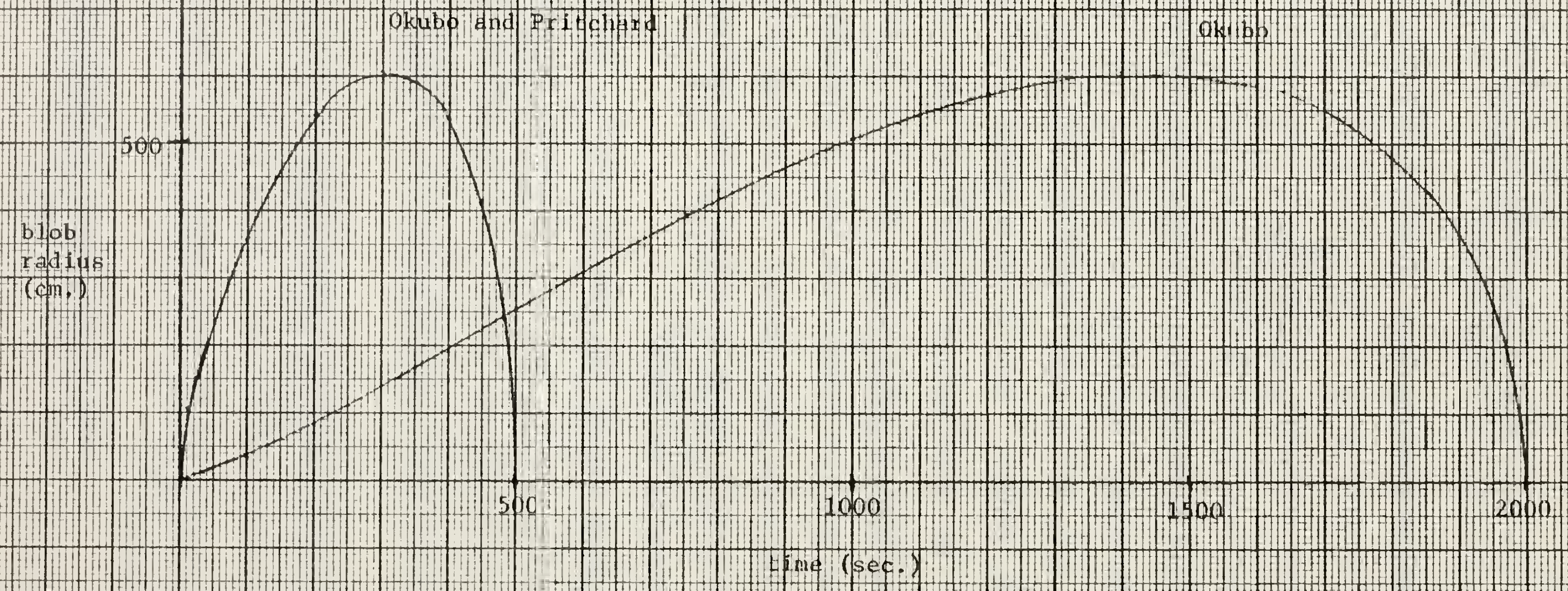

Figure 1

Comparison of blobs for two ocean turbulence models.

$$(12) \qquad 1 = \frac{(4/3)\pi r_0^3}{(2\pi Dt)^{3/2}} \exp\left(-\rho^2(t)/2Dt\right),$$

where r_0 is the radius of the largest sphere within which M units of material would be detectable if spread evenly.

Solving (12) for $\rho^2(t)$, and defining $K = (9\pi/2)^{1/3}$, we have

$$(13) \qquad \rho^2(t) = 2Dt \; \ell n\left(\frac{r_0^2}{KDt}\right)^{3/2} = 3Dt \; \ell n\left(\frac{r_0^2}{KDt}\right),$$

except that $\rho(t) = 0$ for $t > r_0^2/KD$. The rate of detection at time t is now $r(t) = v\left(\pi\rho^2(t)\right)/V$, where V is the volume of the confining region, so

$$(14) \qquad C = \pi v/V \int_0^\infty \rho^2(t)\, dt.$$

Substituting $u = KDt/r_0^2$,

$$(15) \qquad C = (\pi v/V)(r_0^2/KD)(3r_0^2/K) \int_0^1 u \; \ell n(1/u)\, du$$

The definite integral [3] is 1/4, so

$$(16) \qquad C = .1343 \; (vr_0^4/VD).$$

Appendix:

Let $r(x)$ be the detection rate on a blob at a time x after it has been emitted, and suppose blobs are emitted in a Poisson process with rate λ. Let $N(t)$ be the number of blobs emitted in $[0,t]$, and let S_i be the times at which they are emitted, $i = 1, \ldots, N(t)$. For $0 \leq u \leq t$, the total detection rate at time u is $\sum_{i=1}^{N(t)} r(u-S_i)$, where $r(x)$ is taken to be 0 for $x < 0$. Therefore the average number of detections in the interval $[t-\Delta,t]$, given the emission times, is

$$\text{(A1)} \qquad Y(t) = \sum_{i=1}^{N(t)} \int_{t-\Delta}^{t} r(u-S_i)\, du, \quad \Delta \geq 0.$$

The stochastic process $Y(t)$ is a filtered Poisson process with impulse response $h(t,\tau) = \int_{t-\Delta}^{t} r(u-\tau)\, du = \int_{t-\tau-\Delta}^{t-\tau} r(v)\, dv$. The generating function of $Y(t)$ is therefore known [2]:

$$\text{(A2)} \qquad \psi_t(S) \equiv E(\exp(SY(t))) = \exp\left(\lambda \int_0^t [\exp(Sh(t,\tau)) - 1]\, d\tau\right)$$

By substituting $x = t - \tau$, this can be written

$$\text{(A3)} \qquad \psi_t(S) = \exp\left(\lambda \int_0^t [\exp(S \int_{x-\Delta}^{x} r(v)\, dv) - 1]\, dx\right)$$

The number of detections in the interval $[t-\Delta,t]$ is a Poisson random variable with mean $Y(t)$. The probability of no detections in this interval is therefore $E(\exp(-Y(t))) = \psi_t(-1)$. If T is the time from $t - \Delta$ to the next detection, we therefore have $P(T > \Delta) = \psi_t(-1)$. Letting

$t \to \infty$ to obtain equillibrium,

$$\text{(A4)} \qquad P(T > \Delta) = \exp\left\{\lambda \int_0^\infty \left[\exp\left(-\int_{x-\Delta}^{x} r(v)\, dv\right) - 1\right] dx\right\}$$

Since T is a positive random variable, we also have

$$\text{(A5)} \qquad E(T) = \int_0^\infty P(T > \Delta)\, d\Delta,$$

and this provides an explicit formula for $E(T)$ that could be evaluated by numerical integration in any specific case. However, suppose the mean number of contacts per blob (C) is <<1. Since $C = \int_0^\infty r(v)\, dv$, it follows that $\int_{x-\Delta}^{x} r(v)dv$ <<1, and therefore that

$$1 - \exp\left(-\int_{x-\Delta}^{x} r(v)\, dv\right) \approx \int_{x-\Delta}^{\infty} r(v)\, dv,$$ and therefore that

$$\text{(A6)} \qquad P(T > \Delta) \approx \exp(-\lambda\, f(\Delta)), \quad \text{where}$$

$$\text{(A7)} \qquad f(\Delta) = \int_0^\infty \left[\int_{x-\Delta}^{x} r(v)\, dv\right] dx$$

Since $f(0) = 0$, and since $(d/d\Delta)\, f(\Delta) = \int_0^\infty r(x-\Delta)\, dx = C$, it must be true that $f(\Delta) = C\Delta$, and therefore that T is approximately exponential with mean $1/\lambda C$. Furthermore since all of the above approximations can be written as inequalities,

$$\text{(A8)} \qquad E(T) \geq 1/\lambda C,$$

with equality holding when $C << 1$.

References

[1] Larson, Harold J., and B. O. Shubert (1979), Probabilistic Models in Engineering Sciences (vol II), pp 692-693, Wiley, New York.

[2] ibid, sec 7.6.

[3] Standard Math Tables, Chemical Rubber Co.

[4] Pritchard, D. W., et. al (1971), Radioactivity in the Marine Environment (ch. 4), National Academy of Sciences.

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